Grade 9/10 Math Circles
February 8
The Shape of You

Shapes! I quite like shapes. So let's talk about shapes. The following three Math Circles seminars are taken from a grad level course and some of my favorite mathematics. Let's have some fun!

Introduction

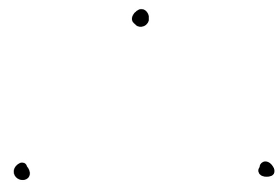
What are some of the most basic shapes you can think of? Observe:

This is a dot.

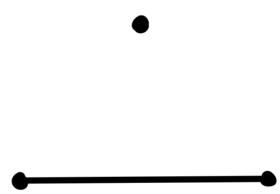
This is two dots.

This is a line. (Hint: a line connects two dots)

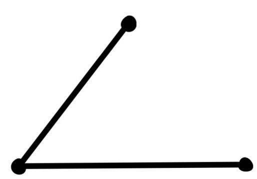
This is three dots (no line).



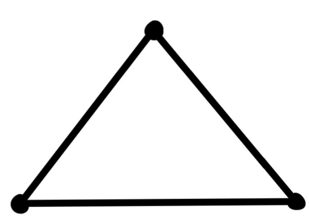
This is a line and a dot (welcome back, line!)



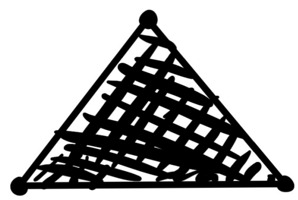
Guess what comes next? That's right, two lines.



We'll keep going, I guess. Three lines?



Drum roll... TRIANGLE

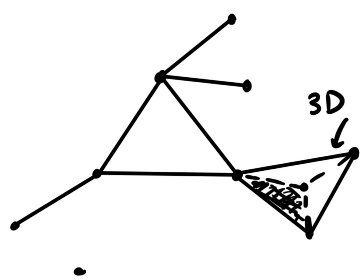


Stop and Think

Can you see the difference between three lines and a triangle? Can you see which shapes "contain" other shapes? Can you think of the next shapes that will come in this pattern?

Fun fact: My four year old nephew can do this, but graduate level students sometimes can't.

Let's think about gluing these together!



Any shape of this form is called a **Simplicial Complex**. For brevity[1], I will refer to these shapes as "simplexes".

[1] Saying "for brevity" in a sentence has never saved anyone any time. Neither has writing footnotes.

Warm-Up

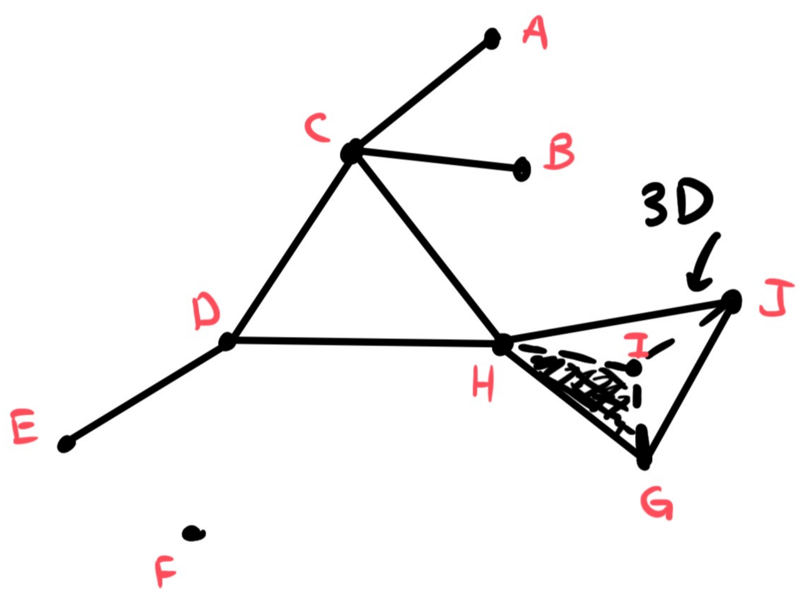
A big part of mathematics[a] is giving different names to different things and calling it productive work. So let's be productive! When communicating ideas about simplexes, it might be useful to have a consistent SYSTEM for writing down any possible shape that is made up of dots, lines, triangles, tetrahedrons, etc. that may be glued together. The SYSTEM that I have implicitely given you here is the easiest one: just draw the shapes!

But how else could you describe simplexes, so that someone else (who understands your SYSTEM) would know exactly what you're talking about, *without any pictures*? In the space below, jot down some ideas for the most efficient, consistent set of rules for a SYSTEM like this that you can think of – you could use words, numbers, brackets, letters, sounds, or any other non-pictoral method of communication. Feel free to discuss ideas with the less stinky person out of the two neighbours on your left and right.

[a] According to a very small minority of mathematicians that I asked.

n-Simplexes

Consider the following simplex:



We need some words to describe this thing. First, the biggest "pieces" are called **facets**. For example, the only facet above is the filled in triangle triangle GHI, since that is the highest dimensional piece in the picture (everything else is either a dot or a line):

In general, the (unglued) "pieces" are called **faces**. Faces can only be dots, lines, triangles, etc. It is helpful to say that a simplex **contains** its faces. For example, the faces above are all the dots A,B,C,D,E,F,G,H,I,J, all the lines, AC,CB,CD,DH,HC,DE,JH,JG,GH,JI,HI,GI, and the triangle GHI. Notice: all facets are faces, but not all faces are facets!

Last, I will also mention the simplex that doesn't contain anything. It looks like this:

It is called the **Irrelevant Complex**.

Theorem: Mathematicians can sometimes be mean.

Proof: They came up with the **Irrelevant Complex** and then called it irrelevant.

Discussion: we've already seen that every simplex contains faces that are "lower-dimensional". For example, a triangle contains the three lines and three dots that we draw in a triangle. Do you think that we should say that an arbitrary complex **contains** the irrelevant simplex? Why or why not?

Now that we've studied simplexes in general, let's consider a natural *family*[2] of simplexes. Consider the simplest n-dimensional shapes (fill in the chart):

Number of Dimensions	Name of Shape	Picture
-1	Irrelevant Simplex (0 simplex)	
0	Dot (1 simplex)	
1	Line (2 simplex)	
2	Triangle (3 simplex)	
3	Tetrahedron (4 simplex)	
4		
5		
6		

[2]In math, a family usually means "a collection of similar-looking *things*". There are no brothers, sisters, or eccentric uncles involved.

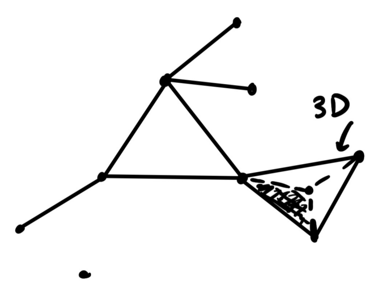
Stop and Think
What's a useful way to define the **dimension** of a complex, so that it matches your intuition?

We define the **dimension of a face** to be the number of dots in the face, minus 1. Does this match our usual idea of dimension?

Bonus: What's the dimension of the **Irrelevant Complex**?

One useful way to write down all the dimensional data of a simplex is a **Dimension Vector**[3], which is just a list of the number of *faces* of a particular dimension in a complex, for each dimension.

For example, taking the following simplex,



We would write its **Dimension Vector** as follows: (1,10,12,1). This corresponds to:

Dimension	-1	0	1	2
Number of Faces	1	10	12	1

Note: Since every simplex has a facet of maximum dimension, if we kept going with our vector, it would just have a bunch of zeroes all in a row. So you can stop counting faces once you've counted all the faces (duh) and omit the zeroes.

Using your n-simplex chart in a previous page, count the number of faces for each dimension and record your results below:

Hint: Try to make your columns line up nicely.

[3]What did I say about giving things funny names in math and calling it productivity??

Which Simplex?	Dimension Vector
-1-simplex	
0-simplex	
1-simplex	
2-simplex	
3-simplex	
4-simplex	
5-simplex	

Stop and Think
Now put your hands in the air and wave 'em around like you just don't care.

Just kidding. Can you spot any patterns in the above numbers? How do the numbers increase or decrease? Why? Can you make a guess for what the next row will look like without drawing and counting the 6-simplex?

See the next page for ***Spoilers***.

Triangles All the Way Down

The chart above is known as **Pascal's Triangle**. Here it is in its full glory:

$$\begin{array}{ccccccccccccccc} &&&&&&&1&&&&&&& \\ &&&&&&1&&1&&&&&& \\ &&&&&1&&2&&1&&&&& \\ &&&&1&&3&&3&&1&&&& \\ &&&1&&4&&6&&4&&1&&& \\ &&1&&5&&10&&10&&5&&1&& \\ &1&&6&&15&&20&&15&&6&&1& \\ 1&&7&&21&&35&&35&&21&&7&&1 \end{array}$$

Let's make some new definitions. For two numbers $0 \le k \le n$, we define a **combination** as the **number of faces of dimension k in an n-simplex**. In other words, this is the (k-1)-th entry in the f-vector for the n-simplex. We denote this number by the symbols

$$\binom{n}{k}$$

Cute Fact

Have you seen this symbol before? On a calculator, you'd see this written as "nCk".

Some other interpretations (can you prove this?) of this number:

$$\begin{aligned}\binom{n}{k} &= \text{the number of faces of dimension } k \text{ in an } n\text{-simplex (this is the definition)} \\ &= \text{the number of ways to include } k \text{ dots to make a } k-1 \text{ dimensional face in an } n\text{-simplex} \\ &= \text{the number of ways to exclude } k \text{ dots to make a } n-k-1 \text{ dimensional face in an } n\text{-simplex}\end{aligned}$$

Let's PROVE our first big results for this Math Circles! We'll need these results for later.

Theorem 1
For all numbers $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof: Notice what this formula represents: it says that to figure out what a number in Pascal's Triangle is, we just add the two numbers above it! (Check the picture of Pascal's Triangle – does this check out?).

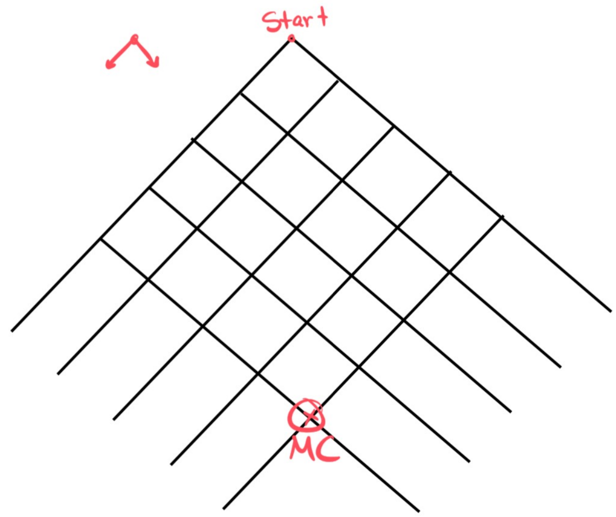
To prove it, we only require to do a bunch of examples of calculating dimension vectors and eventually get lazy with our counting. Suppose we are counting $(k-1)$-dimensional faces in an n-simplex. Let's call one of the dots in the n-simplex D (for dot). Notice that if we ignored D and all the lines that are connected to it, then we get an $(n-1)$-simplex!

So instead of counting randomly, let's count first the number of ways to pick k dots, where we *specifically don't include the dot* D, PLUS the number of ways to pick k dots, where we *specifically do include the dot* D. In the second case, if we have already picked D, then we might as well just focus on picking the other $k-1$ dots in the $(n-1)$-simplex we get by ignoring D. As an exercise, work out the rest of the details on your own!

Hint: if you get stuck, try some examples! How many triangles in a 5-simplex? How many lines in a 6-simplex? How many tetrahedrons in a 7-simplex?

On to Theorem 2!

Here's the set-up. I have a problem. Here's my problem. I'm a lazy student who has to run to my 8:30am lecture on campus. However, the streets in Waterloo look like this:



Being a lazy student, every time I walk a block, I want to get closer to my destination: MC. So I'm always going[4] in the "downwards" direction, but I can choose at each intersection to get left or right.

Theorem 2: Path-Counting
If we overlay the grid above with Pascal's Triangle, then the entry at each intersection is exactly equal to the number of ways I can get[a] there from the top!

Proof: The proof of this will be covered next class, but I encourage you to try it yourself! For additional information, see the Problem Set.

[a] run

This concludes the content for our first lecture! If you'd like to practice some more, see the Problem Set. And come back next week for more fun stuff! If you have any questions or concerns (or cool math), I am available over email at mgusak@uwaterloo.ca. I'm always happy to talk about cool math.

This session touches on lots of math! This is math that you can continue doing in undergraduate courses here at Waterloo. Mostly, the math here is what you'd see in combinatorics (MATH239/MATH249), simplicial homology (PMATH467), and other topics in combinatorics (CO439). Ask me about courses!

[4] running